ON TENSOR CATEGORIES ATTACHED TO CELLS IN AFFINE WEYL GROUPS

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ABSTRACT. This note is devoted to Lusztig's bijection between unipotent conjugacy classes in a simple complex algebraic group and 2-sided cells in the affine Weyl group of the Langlands dual group; and also to the description of the reductive quotient of the centralizer of the unipotent element in terms of convolution of perverse sheaves on affine flag variety of the dual group conjectured by Lusztig in [L4]. Our main tool is a recent construction by Beilinson, Gaitsgory and Kottwitz, the so-called sheaf-theoretic construction of the center of an affine Hecke algebra (see [Ga]). We show how this remarkable construction provides a geometric interpretation of the bijection, and allows to prove the conjecture.

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1. Introduction.

Let G be a split simple algebraic group. Let W be the corresponding affine Weyl group; and let J be the asymptotic affine Hecke algebra [L2]. Recall that J is an algebra over $\mathbb{Z}$, and it comes with a basis t_w parametrzied by W. The group W is the union of two-sided cells [L1], and the algebra J is the direct sum of algebras, $J = \oplus J_{\underline{c}}$, where $\underline{c}$ runs over the set of two-sided cells, and $J_{\underline{c}}$ is the span of t_w, $w \in \underline{c}$. Let also $W_f \subset W$ be the finite Weyl group, and $W^f \subset W$ be the set of minimal length representatives of double cosets $W_f\backslash W/W_f$. Let J^f be the span of t_w, $w \in W^f$, and $J^f_{\underline{c}} = J^f \cap J_c$. It follows from the result of [LX] that $J^f \subset J$, $J^f_{\underline{c}} \subset J_{\underline{c}}$ are subalgebras.

Let ${}^LG$ be the Langlands dual group (over an algebraically closed field of characteristic zero). In [L3] Lusztig constructed a bijection between two-sided cells in W and unipotent conjugacy classes in ${}^LG$; for a 2-sided cell $\underline{c}$ we will denote by $N_{\underline{c}} \in {}^LG$ a representative of the unipotent conjugacy class corresponding to $\underline{c}$. In [L4] the based algebras $J_{\underline{c}}$, $J^f_{\underline{c}}$ are realized as Grothendieck groups of certain semisimple monoidal (tensor without commutativity) categories, which we denote respectively by $A_{\underline{c}}$, $A^f_{\underline{c}}$ (thus $A^f_{\underline{c}}$ is a monoidal subcategory in $A_{\underline{c}}$). Categories $A_{\underline{c}}$, $A^f_{\underline{c}}$ are defined as subcategories of (semisimple) perverse sheaves on the affine flag

manifold of G; the monoidal structure is provided by the truncated convolution, see *loc. cit.*

The following conjectural description of monoidal categories $A_{\underline{c}}$, $A_{\underline{c}}^f$ (and hence of based rings $J_{\underline{c}}$, $J_{\underline{c}}^f$) was proposed in [L4], §3.2. Let $Z = Z_{{}^LG}(N_{\underline{c}})$ be the centralizer of $N_{\underline{c}}$ in ${}^LG$. For any two-sided cell $\underline{c}$ there exists (conjecturally) a finite set X with a Z action, such that $A_{\underline{c}}$ is equivalent to the category $Vect_{ss}^Z(X \times X)$ of semisimple Z-equivariant sheaves on $X \times X$, with monoidal structure given by convolution. The finite set X should contain a preferred point $x \in X$ fixed by Z; thus the category $Vect_{ss}^Z(X \times X)$ contains a monoidal subcategory $Vect_{ss}^Z(\{(x,x)\}) = Rep^{ss}(Z)$ (the category of semisimple representations of Z). This subcategory should be identified with $A_{\underline{c}}^f \subset A_{\underline{c}}$.

In this note we (extend and) prove the part of the above conjecture, which asserts the equivalence $A_{\underline{c}}^f = Rep^{ss}(Z)$. The proof is based on a recent construction by Gaitsgory (following an idea of Beilinson and Kottwitz), see [Ga].

Recall that the link between representations of ${}^LG$ and theory of perverse sheaves on affine Grassmanian of G is provided by the so-called geometric version of the Satake isomorphism (the idea going back to [L0] is developed in [Gi] and [MV]; see also [BD]). The classical Satake isomorphism is an isomorphism between the spherical Hecke algebra $\mathcal{H}_{sph}$ and the ring $\mathcal{R}({}^LG)$, where for an algebraic group H we write $\mathcal{R}(H)$ for its representation ring (Grothendieck group of the category $Rep(H)$ with multiplication provided by the tensor product). Its geometric (or categorical) version is an equivalence of tensor categories between $Rep({}^LG)$ and perverse sheaves on the affine Grassmanian.

Further, a theorem of Bernstein (see e.g. [L0], Proposition 8.6) asserts that the center of the Iwahori-Matsumoto Hecke algebra $\mathcal{H}$ is also isomorphic to $\mathcal{R}({}^LG)$. The main result of [Ga] provides a geometric (categorical) counterpart of this isomorphism. More precisely, it defines an action of the tensor category $Rep({}^LG)$ on the category of perverse sheaves on affine flags; on the level of Grothendieck groups this amounts to the action of the center of $\mathcal{H}$, $Z(\mathcal{H}) = \mathcal{R}({}^LG)$ on $\mathcal{H}$. This action enjoys various favorable properties; it also carries a canonical unipotent automorphism $\mathfrak{M}$ (the monodromy).

The idea of the present note is that one can identify certain subquotient categories of perverse sheaves on affine flags related to a 2-sided cell $\underline{c}$ with $Rep(H)$ for a subgroup $H \subset Z(N_{\underline{c}})$, in such a way that the canonical action of $Rep({}^LG)$ constructed in [Ga] is identified with the tautological action of $Rep({}^LG)$ on $Rep(H)$

$$V : W \mapsto Res_H^{{}^LG}(V) \otimes W;$$

moreover this requirement fixes the identification with $Rep(H)$ uniquely. Then the monodromy automorphism $\mathfrak{M}$ provides (by Tannakian formalism) a unipotent element $N \in {}^LG$, commuting with H. This element lies in the conjugacy class attached to $\underline{c}$ by Lusztig.

With these tools in hand, the Lusztig's conjecture becomes an exercise in Tannakian formalism (at least modulo some powerful Theorems of Lusztig on the structure of asymptotic Hecke algebras).

I want to point out that although this note contains a complete proof of the stated result, the argument presented here may not be optimal. In particular, at several places we use "as a black box" results of Lusztig on asymptotic Hecke algebras to check a categorical property of perverse sheaves on affine flags. In Remarks 4,

7 below we discuss a possible plan for replacing some of these uses by a direct geometric argument (in other words, for providing a geometric proof of Lusztig's results). However, we do not suggest such a geometric proof e.g. for Lusztig's description of the unit element in the asymptotic Hecke algebra (equivalent to certain properties of Duflo involutions).

2. Preliminaries on tensor categories

In this section we collect some general Lemmas on Tannakian formalism. They are probably well-known (anyway, the proof is straightforward) but a reference was not found.

For a category A (full subcategory of a triangulated category) we will write $K(A)$ for the Grothendieck group of A. For an object $X \in A$ we will denote its class by $[X] \in K(A)$.

Lemma 1. *Let $\mathcal{A}$ be an abelian category with all objects having finite length. Let $\otimes$ denote a functor $\mathcal{A} \times \mathcal{A} \to A$ which is linear and mid-exact in each variable. For an object $X \in \mathcal{A}$ let X^{ss} denote the semisimplification of X. Let $X, Y \in \mathcal{A}$ be two objects satisfying*

$$[X \otimes Y] = [X^{ss} \otimes Y^{ss}] \tag{1}$$

where $[X]$ denotes the class of X in the Grothendieck group $K^0(\mathcal{A})$. Then (1) *remains true when X, Y are replaced by a subquotient. Moreover, if X', Y' are subquotients of respectively X, Y and $0 \to X'' \to X' \to X''' \to 0$ is an exact sequence, then the sequence $0 \to X'' \otimes Y' \to X' \otimes Y' \to X''' \otimes Y' \to 0$ is exact.*

Proof For $\alpha, \beta \in K^0(\mathcal{A})$ let us write $\alpha \leq \beta$ if $\beta - \alpha$ is a class of actual (as opposed to virtual) object. Then for any s.e.s. $0 \to X_1 \to X_2 \to X_3 \to 0$ we have

$$[X_2 \otimes Y] \leq [X_1 \otimes Y] + [X_3 \otimes Y] \tag{2}$$

for any Y, with equality being true iff the sequence $0 \to X_1 \otimes Y \to X_2 \otimes Y \to X_3 \otimes Y \to 0$ is exact. Hence the first statement implies the second.

By induction in Jordan-Hoelder series we see that $[X \otimes Y] \leq [X^{ss} \otimes Y^{ss}]$ for any X, Y. Moreover, if for subquotients X', Y' of X, Y the strict inequality $[X' \otimes Y'] < [X'^{ss} \otimes Y'^{ss}]$ holds, then successive use of (2) shows that $[X \otimes Y] < [X^{ss} \otimes Y^{ss}]$, which contradicts (1). $\square$

Definition 1. Let $\mathcal{A}$ be a monoidal category, and $\mathcal{B}$ be a tensor (symmetric monoidal) category. A *central functor* from $\mathcal{B}$ to $\mathcal{A}$ is a monoidal functor $F : \mathcal{B} \to \mathcal{A}$ together with an isomorphism

$$\sigma_{X,Y} : \ F(X) \otimes Y \cong Y \otimes F(X) \tag{3}$$

fixed for all $X \in \mathcal{B}$, $Y \in \mathcal{A}$, subject to the following compatibilities.

i) $\sigma_{X,Y}$ is functorial in X, Y;

ii) For $X, X' \in \mathcal{B}$ the isomorphism $\sigma_{X,F(X')}$ coincides with the composition

$$F(X) \otimes F(X') \cong F(X \otimes X') \cong F(X' \otimes X) \cong F(X') \otimes F(X)$$

(where the middle isomorphism comes from the commutativity constraint in $\mathcal{B}$, and the other two from the tensor structure on F).

iii) For $Y_1, Y_2 \in \mathcal{A}$ and $X \in \mathcal{B}$ the composition

$$F(X) \otimes Y_1 \otimes Y_2 \xrightarrow{\sigma_{X,Y_1} \otimes Y_2} Y_1 \otimes F(X) \otimes Y_2 \xrightarrow{Y_1 \otimes \sigma_{X,Y_2}} Y_1 \otimes Y_2 \otimes F(X)$$

coincides with $\sigma_{X,Y_1\otimes Y_2}$.

iv) For $Y\in\mathcal{A}$ and $X_1,X_2\in\mathcal{B}$ the composition

$$F(X_1\otimes X_2)\otimes Y\cong F(X_1)\otimes F(X_2)\otimes Y \xrightarrow{F(X_1)\otimes\sigma_{X_2,Y}} F(X_1)\otimes Y\otimes F(X_2) \xrightarrow{\sigma_{X_1,Y}\otimes F(X_2)}$$
$$Y\otimes F(X_1)\otimes F(X_2)=Y\otimes F(X_1\otimes X_2)$$

coincides with $\sigma_{X_1\otimes X_2,Y}$.

Remark 1. [1] For a monoidal category $\mathcal{A}$ its *center* $\mathcal{Z}(\mathcal{A})$ is defined as the category of pairs (X,σ), where $X\in\mathcal{A}$, and $\sigma=(\sigma_Y)$ is a collection of isomorphisms $\sigma_Y: X\otimes Y\cong Y\otimes X$, subject to certain compatibilities. Then $\mathcal{Z}(\mathcal{A})$ turns out to carry a natural structure of a braided monoidal category. If $\mathcal{A}$ is the category of representations of a Hopf algebra A, then $\mathcal{Z}(\mathcal{A})$ is identified with the category of representations of Drinfeld's double of A.

One can check that a central functor from a (symmetric) tensor category $\mathcal{B}$ to a monoidal category $\mathcal{A}$ is the same as a monoidal functor from $\mathcal{B}$ to $\mathcal{Z}(\mathcal{A})$, which intertwines commutativity in $\mathcal{B}$ with the braiding in $\mathcal{Z}(\mathcal{A})$.

Proposition 1. *Let k be an algebraically closed field. Let $\mathcal{A}$ be a k-linear abelian monoidal category with a unit object $\mathbb{I}$ such that $End(\mathbb{I})=k$. We assume that the product in $\mathcal{A}$ is exact in each variable.*

Let G be an algebraic group over k, and $Rep(G)$ be the category of its finite dimensional algebraic representations. Let $F: Rep(G)\to\mathcal{A}$ be an exact central functor. Suppose that any object $Y\in\mathcal{A}$ is isomorphic to a subquotient of $F(X)$ for some $X\in Rep(G)$.

Assume also[2] *that k is uncountable and $Hom(X,Y)$ is finite dimensional for $X,Y\in\mathcal{A}$. Then there exists an algebraic subgroup $H\subset G$, and an equivalence of monoidal categories $\Phi: Rep(H)\widetilde{\longrightarrow}\mathcal{A}$, such that*

$$F\cong\Phi\circ Res_H^G.$$

The subgroup $H\subset G$ is defined uniquely up to conjugation.

Proof G acts on itself by left translations, making the space $\mathcal{O}(G)$ of regular functions on G an algebraic G-module, thus an ind-object of $Rep(G)$. Let $\underline{\mathcal{O}}_G$ denote this ind-object. It is a ring ind-object, i.e. we have a multiplication morphism $m:\underline{\mathcal{O}}_G\otimes\underline{\mathcal{O}}_G\to\underline{\mathcal{O}}_G$ satisfying the usual commutative ring axioms. For a ring ind-object ? we will write $m_?$ for the multiplication morphism $m_?: ?\otimes ?\to ?$. Let $\mathcal{J}\subsetneq F(\underline{\mathcal{O}}_G)$ be a maximal left ideal subobject, i.e. $\mathcal{J}$ is a maximal proper ind-subobject in $F(\underline{\mathcal{O}}_G)$ satisfying

$$m_{F(\underline{\mathcal{O}}_G)}(F(\underline{\mathcal{O}}_G)\otimes\mathcal{J})\subset\mathcal{J}. \tag{4}$$

Then $\mathcal{J}$ is also a right ideal subobject, i.e. we have

$$m(\mathcal{J}\otimes F(\underline{\mathcal{O}}_G))\subset\mathcal{J}; \tag{5}$$

indeed, commutativity of $\underline{\mathcal{O}}_G$, and property (ii) in the definition of a central functor show that

$$m_{F(\underline{\mathcal{O}}_G)}\circ\sigma_{\underline{\mathcal{O}}_G,F(\underline{\mathcal{O}}_G)}=m_{F(\underline{\mathcal{O}}_G)},$$

[1]The content of this remark was communicated to me by Drinfeld.

[2]These assumptions are not necessary, and are imposed to shorten the proof.

and property (i) yields the equality

$$m_{F(\underline{\mathcal{O}}_G)}|_{\mathcal{J}\otimes F(\underline{\mathcal{O}}_G)} \circ \sigma_{\underline{\mathcal{O}}_G,\mathcal{J}} = m_{F(\underline{\mathcal{O}}_G)}|_{F(\underline{\mathcal{O}}_G)\otimes \mathcal{J}},$$

which implies (5).

Set $\underline{\mathcal{O}}_H = F(\underline{\mathcal{O}}_G)/\mathcal{J}$. (4), (5) imply that $\underline{\mathcal{O}}_H$ is a ring ind-object of $\mathcal{A}$. Thus the category of $\underline{\mathcal{O}}_H$-module (ind)objects in $\mathcal{A}$ is well-defined. We will denote this category by $\underline{\mathcal{O}}_H - mod$, call its objects $\underline{\mathcal{O}}_H$-modules, and write $Hom_{\underline{\mathcal{O}}_H}$ instead of $Hom_{\underline{\mathcal{O}}_H-mod}$.

Then $\underline{\mathcal{O}}_H - mod$ is an abelian category, and $\underline{\mathcal{O}}_H \in \underline{\mathcal{O}}_H - mod$ is a simple object. Thus $K = End_{\underline{\mathcal{O}}_H}(\underline{\mathcal{O}}_H)$ is a division algebra, and $V \mapsto V \otimes_K \underline{\mathcal{O}}_H$ is an equivalence between (right) finite K-modules and the full subcategory in $\underline{\mathcal{O}}_H - mod$ generated by $\underline{\mathcal{O}}_H$ under finite direct sums and subquotients (we will call such $\underline{\mathcal{O}}_H$-modules free; argument below implies that in fact any $\underline{\mathcal{O}}_H$ module is free).

Lemma 2. *We have $K = Hom_{\mathcal{A}}(\mathbb{I}, \underline{\mathcal{O}}_H) = k$.*

Proof $\underline{\mathcal{O}}_H$ is a unital ring object since $\underline{\mathcal{O}}_G$ is, i.e. the unit $\iota : \mathbb{I} \hookrightarrow \underline{\mathcal{O}}_H$ is fixed. The map $\phi \mapsto \phi \circ \iota$ provides an isomorphism $End_{\underline{\mathcal{O}}_H}(\underline{\mathcal{O}}_H) \cong Hom_{\mathcal{A}}(\mathbb{I}, \underline{\mathcal{O}}_H)$, with inverse isomorphism given by $i \mapsto m_{\underline{\mathcal{O}}_H} \circ (i \otimes Id_{\underline{\mathcal{O}}_H})$. Thus the first equality is clear.

To check the second one, notice that cardinality of a basis of a division algebra K over an algebraically closed field k is not less than the cardinality of k (indeed, for $x \in K$, $x \notin k$ the elements $(x-\lambda)^{-1}$, $\lambda \in k$ are linearly independent). However, $\underline{\mathcal{O}}_G$ is a countable union of objects of $Rep(G)$, hence $Hom_{\mathcal{A}}(\mathbb{I}, F(\underline{\mathcal{O}}_G))$ is at most countable dimensional. $\square$

Corollary 1. *a) For any $X \in \mathcal{A}$ we have $X \otimes \underline{\mathcal{O}}_H \cong V \otimes \underline{\mathcal{O}}_H$ for some finite dimensional k-vector space V.*

b) The functor $\Phi_H : \mathcal{A} \to Vect$ given by $\Phi_H : X \mapsto Hom(\mathbb{I}, X \otimes \underline{\mathcal{O}}_H)$ is exact, admits a structure of a monoidal functor, and we have a canonical isomorphism of monoidal functors

$$\Phi_G \cong \Phi_H \circ F, \tag{6}$$

where $\Phi_G : Rep(G) \to Vect$ is the fiber functor.

Proof a) Let first $X = F(Y)$ for $Y \in Rep(G)$. We have an isomorphism of $\underline{\mathcal{O}}_G$-modules $Y \otimes \underline{\mathcal{O}}_G \cong \Phi_G(Y) \otimes \underline{\mathcal{O}}_G$, hence an isomorphism of $F(\underline{\mathcal{O}}_G)$-modules

$$X \otimes F(\underline{\mathcal{O}}_G) \cong \Phi_G(Y) \otimes F(\underline{\mathcal{O}}_G).$$

Replacing each side of the last equality by the maximal quotient on which $F(\underline{\mathcal{O}}_G)$ acts through $\underline{\mathcal{O}}_H$ we get an isomorphism of $\underline{\mathcal{O}}_H$-modules:

$$F(Y) \otimes \underline{\mathcal{O}}_H \cong \Phi_G(Y) \otimes \underline{\mathcal{O}}_H.$$

Since any $X \in \mathcal{A}$ is a subquotient of $F(Y)$ for some $Y \in Rep(G)$ we see that the $\underline{\mathcal{O}}_H$-module $X \otimes \underline{\mathcal{O}}_H \in \underline{\mathcal{O}}_H - mod$ is a subquotient of the free $\underline{\mathcal{O}}_H$-module $\Phi_G(Y) \otimes \underline{\mathcal{O}}_H$, hence is also free (i.e. has the form $V \otimes \underline{\mathcal{O}}_H$ for a k-vector space V).

Proof of (b). Notice that (a) together with Lemma 2 imply that $X \otimes \underline{\mathcal{O}}_H = \Phi_H(X) \otimes \underline{\mathcal{O}}_H$ canonically for $X \in \mathcal{A}$. This shows exactness of Φ_H, and also establishes monoidal structure on Φ_H, for we have

$$X \otimes Y \otimes \underline{\mathcal{O}}_H = X \otimes (\Phi_H(Y) \otimes \underline{\mathcal{O}}_H) = \Phi_H(X) \otimes \Phi_H(Y) \otimes \underline{\mathcal{O}}_H.$$

Finally, the isomorphism

$$X \otimes \underline{\mathcal{O}}_G \cong \Phi_G(X) \otimes \underline{\mathcal{O}}_G$$

for $X \in Rep(G)$ yields (by applying F, and taking the maximal quotient on which $\underline{\mathcal{O}}_G$ acts through $\underline{\mathcal{O}}_H$) an isomorphism

$$F(X) \otimes \underline{\mathcal{O}}_H \cong \Phi_G(X) \otimes \underline{\mathcal{O}}_H,$$

hence an isomorphism (6). □

We can now finish the proof of Proposition 1. According to §2 of [DM], a functor Φ_H as in Corollary 1 (b) above yields a bialgebra A, an equivalence of monoidal categories $\Psi_H : Comod_A \cong \mathcal{A}$, a morphism of bialgebras $\phi : \mathcal{O}(G) \to A$ and an isomorphism of monoidal functors

$$F \cong \Psi_H \circ \phi_*.$$

Since any object of $\mathcal{A}$ is a subquotient of $F(X)$ for some $X \in Rep(G)$, the morphism $\phi : \mathcal{O}(G) \to A$ is surjective. Thus $A = \mathcal{O}(H)$ for a Zarisski closed subsemigroup $H \subset G$. Thus Proposition 1 follows from the next Lemma. □

We formulate the Lemma in a slightly greater generality than needed for our application (since the proof is the same).

Lemma 3. *Let G be a group scheme of finite type over a Noetherian ring k. Then a closed subsemigroup scheme $H \subset G$ is a subgroup scheme.*

Proof [3] We should check that for any commutative k-algebra R the subsemigroup $H(R) \subset G(R)$ is a subgroup. It is enough to check this for R of finite type over k. For $g \in H(R)$ let $L_g : G_R \to G_R$ be the (left) multiplication by g (here the subindex $_R$ denotes base change to R). Then $L_g(H_R) \subseteq H_R$; we have to check that in fact $L_g(H_R) = H_R$. But otherwise $H_R \supsetneq L_g(H_R) \supsetneq L_g^2(H_R) \supsetneq \dots$ is an infinite decreasing chain of closed subschemes in G_R, which constradicts the fact that G_R is Noetherian. □

Remark 2. In this Remark we outline an alternative argument, which is shorter than the proof of Proposition 1 presented above, but uses a deep Theorem of Deligne [De2], and proves a weaker statement.

In the situation of Proposition 1 assume that $char(k) = 0$, and also that rigidity on the target category $\mathcal{A}$ is given.[4] (In view of Remark 3 below this weaker statement is sufficient for our application.) Then one can show first that there exists a unique commutativity constraint on $\mathcal{A}$ compatible with one in $\mathcal{B}$; thus $\mathcal{B}$ is a Tannakian category. Now a Theorem of Deligne ([De2] Theorem 7.1) says that for an algebraically closed field k of characteristic 0, a k-linear Tannakian category $\mathcal{A}$ admits a fiber functor, and is identified with the category of representations of an algebraic group, provided for any object $X \in \mathcal{A}$ we have $\Lambda^n(X) = 0$ for large n (where Λ^n stands the n-th exterior power). If an object Y is a subquotient of X, then $\Lambda^n(Y)$ is a subquotient of $\Lambda^n(X)$, in particular $\Lambda^n(X) = 0 \Rightarrow \Lambda^n(Y) = 0$. Thus $\mathcal{A}$ satisfies conditions of Deligne's Theorem, and hence $\mathcal{A} = Rep(H)$ for an algebraic group H by that Theorem. The tensor functor $F : Rep(G) \to \mathcal{A} = Rep(H)$ yields by Tannakian formalism a homomorphism $H \to G$; since any object of $Rep(H)$ is a subquotient of $F(X)$ for $X \in Rep(G)$, this homomorphism is injective.

[3] I thank Dima Arinkin, to whom this proof is due.

[4] We do not know whether it is true that for an abelian tensor category with exact tensor product a subquotient of a rigid object of finite length is rigid. If the answer to this question is positive, we can drop here the assumption that rigidity on $\mathcal{A}$ is given.

3. Notations and recollections

3.1. **General notations.** Let G be a split simple algebraic group over $\mathbb{Z}$. Let $F = \mathbb{F}_q((t))$ be a local field of prime characteristic, and $O = \mathbb{F}_q[[t]]$ be its ring of integers.[5] Let $I \subset G(O) \subset G(F)$ be the Iwahori subgroup. There exist canonically defined group schemes $\mathbf{K}$, $\mathbf{I}$ over $\mathbb{F}_q$ (of infinite type) such that $\mathbf{K}(\mathbb{F}_q) = G(O)$, $\mathbf{I}(\mathbb{F}_q) = I$; and a canonical ind-group scheme $\mathbf{G}$ with $\mathbf{G}(\mathbb{F}_q) = G(F)$. We also have ind-varieties $\mathcal{F}\ell = \mathbf{G}/\mathbf{I}$ and $\mathcal{G}\mathfrak{r} = \mathbf{G}/\mathbf{K}$. More precisely, $\mathcal{F}\ell$, $\mathcal{G}\mathfrak{r}$ are direct limits of projective varieties with transition maps being closed imbeddings, and $\mathcal{F}\ell(\mathbb{F}_q) = G(F)/I$, $\mathcal{G}\mathfrak{r}(\mathbb{F}_q) = G(F)/G(O)$.

The orbits of $\mathbf{I}$ on $\mathcal{F}\ell$, $\mathcal{G}\mathfrak{r}$ are finite dimensional, and isomorphic to an affine space; they are sometimes called Schubert cells. As before, W_f is the Weyl group of G, and W is its affine Weyl group. Then W is identified with the set of Schubert cells in $\mathcal{F}\ell$. For $w \in W$ (respectively $w \in W/W_f$) let $\mathcal{F}\ell_w$, $\mathcal{G}\mathfrak{r}_w$ be the corresponding Schubert cells.

Our main character is the derived category of l-adic sheaves on $\mathcal{F}\ell_{\overline{\mathbb{F}}_q}$, constant along the stratification by Schubert cells; here the subscript $_{\overline{\mathbb{F}}_q}$ denotes extension of scalars from $\mathbb{F}_q$ to the algebraic closure $\overline{\mathbb{F}}_q$. More precisely, let $D = D(\mathcal{F}\ell)$ be the full subcategory in the derived category of l-adic sheaves on $\mathcal{F}\ell_{\overline{\mathbb{F}}_q}$ constant along Schubert cells. Let $\mathcal{P} = \mathcal{P}(\mathcal{F}\ell) \subset D(\mathcal{F}\ell)$ be the abelian category of perverse sheaves. It is well known that $D^b(\mathcal{P}) \xrightarrow{\sim} D(\mathcal{F}\ell)$ (see e.g. [BGS], Lemma 4.4.6; we will not use this fact below).

Let $D^I = D^I(\mathcal{F}\ell)$ be the category of $\mathbf{I}$-equivariant l-adic sheaves on $\mathcal{F}\ell$, and $\mathcal{P}^I(\mathcal{F}\ell) \subset D^I$ be the full subcategory of perverse sheaves. Then the functor of forgetting the equivarinat structure $\mathcal{P}^I(\mathcal{F}\ell) \to \mathcal{P}(\mathcal{F}\ell)$ is known to be a full imbedding (though the one from D^I to D is not). Thus $\mathcal{P}^I$ is an abelian subcategory of $\mathcal{P}$, which is not closed under extensions. The convolution product, which we denote by $*$ defines functors $D \times D^I \to D$, and $D^I \times D^I \to D^I$.

3.2. **Central sheaves.** Let $\mathcal{P}_{\mathcal{G}\mathfrak{r}}$ be the category of $\mathbf{K}$ equivariant perverse sheaves on $\mathcal{G}\mathfrak{r}$. It is known ([L0], see also [Ga] for an alternative proof and a generalization) that for $X, Y \in \mathcal{P}_{\mathcal{G}\mathfrak{r}}$ the convolution $X * Y \in \mathcal{P}_{\mathcal{G}\mathfrak{r}}$. Further, convolution endows $\mathcal{P}_{\mathcal{G}\mathfrak{r}}$ with the structure of a monoidal category, and it naturally extends to a structure of a commutative rigid tensor category with a fiber functor; the resulting Tannakian category is equivalent to the category $Rep({}^LG)$ of algebraic representation of the Langlands dual group ${}^LG$ over $\overline{\mathbb{Q}_l}$ (see [Gi], [MV], at least for an analogous statement over $\mathbb{C}$, and also [BD]). We will identify $Rep({}^LG)$ with $\mathcal{P}_{\mathcal{G}\mathfrak{r}}$.

In [Ga] a functor $Z : Rep({}^LG) = \mathcal{P}_{\mathcal{G}\mathfrak{r}} \to P^I(\mathcal{F}\ell)$ was constructed. It enjoys the following properties.

i) $\pi_* \circ Z \cong id$, where $\pi : \mathcal{F}\ell \to \mathcal{G}\mathfrak{r}$ is the projection.

ii) (Exactness of convolution) For $\mathcal{F} \in \mathcal{P}_{\mathcal{G}\mathfrak{r}}$, $\mathcal{G} \in \mathcal{P}$ we have $\mathcal{G} * Z(\mathcal{F}) \in \mathcal{P}$.

iii) (Compatibility with convolution and centrality) Z is a central functor from the tensor category $\mathcal{P}_{\mathcal{G}\mathfrak{r}}$ to the monoidal category D^I in the sense of definition 1 above.[6]

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[5] As usual one could replace $\mathbb{F}_q$ by an algebraically closed field of characteristic zero; then we would have to work with Hodge D-modules instead of Weil sheaves in the proof of Lemma 6 below.

[6] Compatibilities (ii), (iii) of the Definition 1 above are not checked in [Ga], but nevertheless hold [Ga1].

iv) (Monodromy) A unipotent automorphism $\mathfrak{M}$ of the functor Z is given, $\mathfrak{M}_{Z(\mathcal{F})} \in Aut(Z(\mathcal{F}))$; it is called the monodromy automorphism (for reasons explained in [Ga]). It satisfies

$$\mathfrak{M}_{Z(\mathcal{F}*\mathcal{F}')} = \mathfrak{M}_{Z(\mathcal{F})} * \mathfrak{M}_{Z(\mathcal{F}')}, \tag{7}$$

where we identified $End(Z(\mathcal{F}*\mathcal{F}')) = End(Z(\mathcal{F}) * Z(\mathcal{F}'))$ by means of (iii).

3.3. **Serre quotient categories.** The set of isomorphism classes of irreducible objects in $\mathcal{P}$ is in bijection with W. For $w \in W$ let L_w be the corresponding irreducible object; more presicely, we set $L_w = j_{w!*}(\overline{\mathbb{Q}_l}[\dim \mathcal{F}\ell_w])$, where j_w denotes the imbedding $\mathcal{F}\ell_w \hookrightarrow \mathcal{F}\ell$, and $\overline{\mathbb{Q}_l}$ is the constant sheaf.

Recall that a Serre subcategory in an abelian category is a strictly full abelian subcategory closed under extensions and subquotients. If A is an abelian category, and B is a Serre subcategory, then the *Serre quotient* A/B is again an abelian category. If every object in A has finite length, then B is uniquely specified by the set of (isomorphism classes of) irreducible objects of A, which lie in B.

Let $\underline{c} \subset W$ be a two-sided cell. Set $W_{<\underline{c}} = \bigcup\limits_{\underline{c}'<\underline{c}} \underline{c}'$; $W_{\leq\underline{c}} = \bigcup\limits_{\underline{c}'\leq\underline{c}} \underline{c}'$; here $\leq$ is the standard partial order on the set of 2-sided cells (see [L1]), and we write $\underline{c}' < \underline{c}$ instead of $\underline{c}' \leq \underline{c}$ & $\underline{c}' \neq \underline{c}$.

For a subset $S \subset W$ let $\mathcal{P}_S$ denote the Serre subcategory of $\mathcal{P}$ whose set of (isomorphism classes) of irreducible objects equals L_w, $w \in S$. We abbreviate $\mathcal{P}_{\leq\underline{c}} = \mathcal{P}_{W_{\leq\underline{c}}}$, $\mathcal{P}_{<\underline{c}} = \mathcal{P}_{W_{<\underline{c}}}$. We also set $P^I_S = P^I \cap P_S$ etc.

Let $\mathcal{P}^I_{\underline{c}}$ denote the Serre quotient category $\mathcal{P}^I_{\leq\underline{c}}/\mathcal{P}^I_{<\underline{c}}$.

4. Truncated convolution categories

4.1. **Truncated convolution and action of the central sheaves.** Let $D_{\leq\underline{c}}(\mathcal{F}\ell)$, $D_{<\underline{c}}(\mathcal{F}\ell)$ be the full triangulated subcategories of $D(\mathcal{F}\ell)$ consisting of complexes with cohomology in, respectively, $\mathcal{P}_{\leq\underline{c}}$, $\mathcal{P}_{<\underline{c}}$. From the definition of a two-sided cell it follows that $D_{\leq\underline{c}}$, $D_{<\underline{c}}$ are stable under convolution with any object of D^I, i.e. $X \in D_{\leq\underline{c}}$, $Y \in D^I \Rightarrow X * Y \in D_{\leq\underline{c}}$, and same for $D_{<\underline{c}}$.

In particular, for $X, Y \in \mathcal{P}^I_{\leq\underline{c}}$ we have $H^i(X*Y) \in \mathcal{P}^I_{\leq\underline{c}}$, and the image of $H^i(X*Y)$ in $\mathcal{P}^I_{\underline{c}}$ depends canonically only on the image of X, Y in $\mathcal{P}^I_{\underline{c}}$. Thus the formula

$$(X,Y) \mapsto H^i(X*Y) \mod \mathcal{P}^I_{<\underline{c}}$$

defines a bilinear functor $\mathcal{P}^I_{\underline{c}} \times \mathcal{P}^I_{\underline{c}} \to \mathcal{P}^I_{\underline{c}}$.

Recall that for a two-cided cell $\underline{c}$ a non-negative integer $a(\underline{c})$ is defined (see [L1]); and we have $H^i(X*Y) \in \mathcal{P}^I_{<\underline{c}}$ for $i > a(\underline{c})$, $X \in P^I_{\leq\underline{c}}$, $Y \in \mathcal{P}^I$.

For $X, Y \in \mathcal{P}^I_{\underline{c}}$ we define their truncated convolution $X \bullet Y \in \mathcal{P}^I_c$ by $X \bullet Y = H^{a(\underline{c})}(X*Y) \mod \mathcal{P}^I_{<\underline{c}}$. For semisimple X, Y this coincides with the definition in [L4].

Also, for $\mathcal{F} \in \mathcal{P}_{\mathcal{G}_\mathfrak{r}}$ the exact functor $\mathcal{G} \mapsto \mathcal{G} * Z(\mathcal{F})$ preserves $\mathcal{P}^I_{\leq\underline{c}}$ and $\mathcal{P}^I_{<\underline{c}}$, hence induces an exact functor from $\mathcal{P}_{\underline{c}}$ to itself (denoted again by $\mathcal{G} \mapsto \mathcal{G} * Z(\mathcal{F})$).

4.2. **Monoidal category $\mathcal{A}_{\underline{c}}$.**

Proposition 2. *Let $\mathcal{A}_{\underline{c}}$ be the strictly full subcategory of $\mathcal{P}^I_{\underline{c}}$ consisting of all (objects isomorphic to) subquotients of $L_w * Z(\mathcal{F})$, $w \in \underline{c}$, $\mathcal{F} \in \mathcal{P}_{\mathcal{G}_\mathfrak{r}}$. Then*

a) Restriction of $\bullet$ to $\mathcal{A}_{\underline{c}} \times \mathcal{A}_{\underline{c}}$ takes values in $\mathcal{A}_{\underline{c}}$, and is exact in each variable.

b) It equips $\mathcal{A}_{\underline{c}}$ with a structure of a monoidal category; the object $\mathbb{I}_{\underline{c}} = \bigoplus_d L_d$, where d runs over the set of Duflo involutions in $\underline{c}$, is a unit object of $(\mathcal{A}_{\underline{c}}, \bullet)$.

Remark 3. It seems possible to check that the monoidal category $\mathcal{A}_{\underline{c}}$ is rigid, i.e. for $X \in \mathcal{A}_{\underline{c}}$ the "dual" object $X\check{}$ is defined together with morphsisms $ev : X \bullet X\check{} \to \mathbb{I}_{\underline{c}}$, and $\delta : \mathbb{I}_{\underline{c}} \to X\check{} \bullet X$ satisfying the usual compatibilities (see e.g. [De2] 2.1.2). Here $X\check{}$ has the following geometric interpretation. The category $\mathcal{P}^I$ has a canonical anti-involution $\iota : \mathcal{P}^I \to (\mathcal{P}^I)^{op}$ induced (loosely speaking) by the morphism $i : \mathbf{G} \to \mathbf{G}$, $i : g \mapsto g^{-1}$. Then

$$X\check{} \cong \mathbb{V}(\iota(X)) \tag{8}$$

canonically, where $\mathbb{V}$ stands for Verdier duality. (We neither use nor prove this fact here).

Proof of the Proposition. a) It follows from the definitions that $\bullet$ is right exact in each variable. We will deduce that it is exact on $\mathcal{A}_{\underline{c}}$ from a result of Lusztig on asymptotic Hecke algebras. (The argument will use this result of Lusztig and mid-exactness of $\bullet$, but not its right exactness; see also Remark 4 below).

The group $K(\mathcal{P}_{\underline{c}})$ has a natural structure of an associative algebra; the product on $K(\mathcal{A}_{\underline{c}})$ is denoted by $\bullet$ and is defined by

$$[L_{w_1}] \bullet [L_{w_2}] = [L_{w_1} \bullet L_{w_2}]$$

for irreducible objects L_{w_1}, $L_{w_2} \in \mathcal{A}_{\underline{c}}$. Thus $K(\mathcal{A}_{\underline{c}}), \bullet$ is the asymptotic Hecke algebra $J_{\underline{c}}$, cf. [L4]. The classes of simple objects $L_w \in \mathcal{P}_{\underline{c}}$ form a standard basis of this algebra; the standard notation for elements of this basis is $t_w = [L_w] \in J_{\underline{c}} = K(\mathcal{P}_{\underline{c}})$.

The key step is the next

Lemma 4. *For $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{P}_{\mathcal{G}\mathfrak{r}}$ and simple objects L_{w_1}, $L_{w_2} \in \mathcal{A}_{\underline{c}}$ we have*

$$[(Z(\mathcal{G}_1) * L_{w_1}) \bullet (L_{w_2} * Z(\mathcal{G}_2))] = [Z(\mathcal{G}_1) * L_{w_1}] \bullet [L_{w_2} * Z(\mathcal{G}_2)]. \tag{9}$$

Proof The Lemma is a consequence of [L2], 2.4. Let us recall the statement of *loc. cit.*

To formulate it we need some notations. Set $A = \mathbb{Z}[v, v^{-1}]$; thus the affine Hecke algebra $\mathcal{H}$ is an A-algebra. Let $C_w \in \mathcal{H}$, $w \in W$ be the Kazhdan Lusztig basis of $\mathcal{H}$. Let also $\mathcal{H}_{\leq \underline{c}}$, $\mathcal{H}_{<\underline{c}}$ be A submodules in $\mathcal{H}$ generated by C_w with w running over the corresponding subset of W. According to the definition of a two-sided cell the A-submodules $\mathcal{H}_{\leq \underline{c}}$, $\mathcal{H}_{<\underline{c}}$ are two-sided ideals in $\mathcal{H}$. Thus $\mathcal{H}_{\underline{c}} := \mathcal{H}_{\leq \underline{c}}/\mathcal{H}_{<\underline{c}}$ is a bimodule over $\mathcal{H}$. Let $S : \mathcal{H}_{\underline{c}} \to J_c \otimes_{\mathbb{Z}} A$ be the isomorphism of A-modules, which sends C_w into $t_w = [L_w] \in J_{\underline{c}} \subset J_{\underline{c}} \otimes A$. Let us transport the structure of an $\mathcal{H}$-bimodule to $J_{\underline{c}} \otimes A$ by means of S; denote the resulting action by $h_1 \otimes h_2 : x \mapsto h_1 * x * h_2$, $h_1, h_2 \in \mathcal{H}$, $x \in J_{\underline{c}} \otimes A$.

Fact 1. *([L2], 2.4) a) Thus defined right (respectively, left) action of $\mathcal{H}$ on $J_{\underline{c}} \otimes A$ commutes with the canonical left (resp., right) action of $J_{\underline{c}}$ on $J_{\underline{c}} \otimes A$. In other words, extending the $\bullet$ product to an A-algebra structure on $J_{\underline{c}} \otimes A$, we get*

$$h_1 * (x \bullet y) * h_2 = (h_1 * x) \bullet (y * h_2). \tag{10}$$

b) The map $\phi_{\underline{c}} : \mathcal{H} \to J_{\underline{c}} \otimes A$ defined by

$$\phi_{\underline{c}}(h) = h * (\sum t_d), \tag{11}$$

(where d runs over the set of Duflo involutions in $\underline{c}$) is an algebra homomorphism. $\square$

We deduce Lemma 4 from (10). In fact we will use only specialization of (10) at $v = 1$, and only for h in the center of $\mathcal{H}$.

Fix a homomorphism $A \to \mathbb{Z}$ sending v to 1, and set $H = \mathcal{H} \otimes_A \mathbb{Z} \cong \mathbb{Z}[W]$. The structure of $\mathcal{H}$-bimodule on $J_{\underline{c}} \otimes A$ yields a structure of H-bimodule on $J_{\underline{c}}$, which we also denote by $h_1 \otimes h_2 : x \mapsto h_1 * x * h_2$. Thus (10) holds for $x, y \in J_{\underline{c}}$ and $h_1, h_2 \in H$.

We have a standard isomorphism $K(\mathcal{P}) = K(\mathcal{P}^I) = K(D^I) \cong H = \mathbb{Z}[W]$ compatible with the ring structure, where the product in $K(D^I)$ is defined by means of convolution: $[X] * [Y] = [X * Y]$.

Let $x = t_{w_1} = [L_{w_1}]$, $y = t_{w_2} = [L_{w_2}] \in K(\mathcal{P}^I_{\underline{c}}) = J_{\underline{c}}$; and $h_i = [Z(\mathcal{G}_i)] \in K(\mathcal{P}^I) = H$, $i = 1, 2$; then we claim that the left-hand side of (9) equals the left-hand side of (10), and the right-hand side of (9) equals the right-hand side of (10).

Indeed, the statement about the right-hand sides follows from the definitions.

To check the statement about the left-hand sides we rewrite

$$(Z(\mathcal{G}_1) * L_{w_1}) \bullet (L_{w_2} * Z(\mathcal{G}_2)) = H^a(Z(\mathcal{G}_1) * L_{w_1} * L_{w_2} * Z(\mathcal{G}_2)) \mod \mathcal{P}^I_{<\underline{c}} =$$
$$H^a(Z(\mathcal{G}_1) * (L_{w_1} * L_{w_2}) * Z(\mathcal{G}_2)) \mod \mathcal{P}^I_{<\underline{c}}.$$

By 3.2 (ii) above, convolution with $Z(\mathcal{G})$ is exact for $\mathcal{G} \in \mathcal{P}_{\mathcal{G}\mathfrak{r}}$, thus the right hand side in the last equality equals $Z(\mathcal{G}_1) * H^a(L_{w_1} * L_{w_2}) * Z(\mathcal{G}_2)$, hence its class equals $h_1 * (t_{w_1} \bullet t_{w_2}) * h_2$. $\square$

Remark 4. It may be possible to get an alternative proof of exactness of $\bullet$ product, not appealing to Lusztig's result (10), by establishing rigidity in $\mathcal{A}_{\underline{c}}$ by direct geometric considerations as in Remark 3 above. (Recall that for an abelian tensor category rigidity implies exactness of tensor product, cf. [DM], Proposition 1.16.)

The last Lemmas together with Lemma 1 imply exactness of $\bullet|_{\mathcal{A}_{\underline{c}} \times \mathcal{A}_{\underline{c}}}$ in each variable, and part (a) of the Proposition.

b) Associativity of truncated convolution follows from associativity of convolution, and equality $H^i(X * Y) = 0 \mod \mathcal{P}^I_{<\underline{c}}$ for $i > a(\underline{c})$, $X, Y \in \mathcal{P}_{\leq \underline{c}}$, because

$$(X \bullet Y) \bullet Z \cong H^{2a}(X * Y * Z) \mod \mathcal{P}^I_{<\underline{c}} \cong X \bullet (Y \bullet Z)$$

canonically. Also, the properties of the functor Z imply that

$$(L_{w_1} * Z(\mathcal{G}_1)) \bullet (L_{w_2} * Z(\mathcal{G}_2)) \cong (L_{w_1} \bullet L_{w_2}) * Z(\mathcal{G}_1 * \mathcal{G}_2)$$

canonically. The right hand side of the last equality lies in $\mathcal{A}_{\underline{c}}$; also, exactness of $\bullet|_{\mathcal{A}_{\underline{c}} \times \mathcal{A}_{\underline{c}}}$ implies that if $X_i \in \mathcal{P}^I_{\underline{c}}$ is a subquotient of $L_{w_i} * Z(\mathcal{G}_i)$, $i = 1, 2$, then $X_1 \bullet X_2$ is a subquotient of $(L_{w_1} \bullet L_{w_2}) * Z(\mathcal{G}_1 * \mathcal{G}_2)$. Thus $\mathcal{A}_{\underline{c}}$ is stable under the $\bullet$-product. Let us check that $\mathbb{I}_{\underline{c}}$ is a unit object in $\mathcal{A}_{\underline{c}}$. Lusztig proved (see [L4], 2.9) that

$$\mathbb{I}_{\underline{c}} \bullet \mathbb{I}_{\underline{c}} \cong \mathbb{I}_{\underline{c}}; \tag{12}$$

$$\mathbb{I}_{\underline{c}} \bullet L_w \cong L_w, \qquad w \in \underline{c}. \tag{13}$$

Given (12) we have only to show that $X \mapsto \mathbb{I}_{\underline{c}} \bullet X$ is an auto-equivalence of $\mathcal{A}_{\underline{c}}$ (cf definition of a unit object of a monoidal category on p. 105 of [DM]). This functor

is injective on morphisms, since it is exact and kills no irreducible objects by (13). Thus to prove it is an equivalence it suffices to construct for all X an isomorphism

$$\mathbb{I}_{\underline{c}} \bullet X \cong X, \tag{14}$$

because then $X \mapsto \mathbb{I}_{\underline{c}} \bullet X$ is surjective on isomorphism classes of objects; and also the map $Hom(X,Y) \to Hom(I \bullet X, I \bullet Y)$ is an an injective map between vector spaces of the same dimension, hence an isomorphism. (In fact, the isomorphism (14) which will be constructed satisfies the natural compatibilities, but we neither check nor use this fact).

We now construct (14). We know it exists for semisimple X by (13). It follows readily that it is also true for $X = L * Z(\mathcal{G})$, where L is semisimple, and $\mathcal{G} \in \mathcal{P}_{\mathcal{G}\mathfrak{r}}$; indeed, we have

$$\mathbb{I}_{\underline{c}} \bullet (L * Z(\mathcal{G})) \cong (\mathbb{I}_{\underline{c}} \bullet L) * Z(\mathcal{G}) \cong L_w * Z(\mathcal{G}). \tag{15}$$

Hence it suffices to check the following. Suppose that $\iota : Y \hookrightarrow L*Z(\mathcal{G})$ is a subobject (where L is semisimple). We need to see that images of the two imbeddings

$$Y \overset{\iota}{\hookrightarrow} L * Z(\mathcal{G}) \overset{(15)}{=} \mathbb{I}_{\underline{c}} \bullet (L * Z(\mathcal{G}));$$

$$\mathbb{I}_{\underline{c}} \bullet Y \overset{\mathbb{I}_{\underline{c}}\bullet\iota}{\hookrightarrow} \mathbb{I}_{\underline{c}} \bullet (L * Z(\mathcal{G}))$$

coincide. Since the functor $? \mapsto \mathbb{I}_{\underline{c}} \bullet ?$ is injective on morphisms, it is enough to ensure coincidence of two subobjects of $\mathbb{I}_{\underline{c}} \bullet (\mathbb{I}_{\underline{c}} \bullet (L * Z(\mathcal{G})))$: image of $\mathbb{I}_{\underline{c}} \bullet \iota$ and image of $\mathbb{I}_{\underline{c}} \bullet (\mathbb{I}_{\underline{c}} \bullet \iota)$. The latter follows from associativity of $\bullet$. $\square$

4.3. **Tannakian category $\mathcal{A}_d$.** Let $d \in \underline{c}$ be a Duflo involution. Then by [L4], 2.9 we have

$$L_d \bullet L_d \cong L_d. \tag{16}$$

Let $\mathcal{A}_d \subset \mathcal{A}_{\underline{c}} \subset \mathcal{P}^I_{\underline{c}}$ be the strictly full subcategory consisting of all (objects isomorphic to) subquotients of $L_d * Z(\mathcal{F})$, $\mathcal{F} \in \mathcal{P}_{\mathcal{G}\mathfrak{r}}$. Let a functor $Res_d : Rep({}^L G) = \mathcal{P}_{\mathcal{G}\mathfrak{r}} \to \mathcal{A}_d$ be defined by $Res_d(\mathcal{G}) = L_d * Z(\mathcal{G})$.

Lemma 5. *a) $\mathcal{A}_d \subset \mathcal{A}_{\underline{c}}$ is a monoidal subcategory, and $L_d \subset \mathcal{A}_d$ is a unit object.*

b) Res_d has a natural structure of a central functor.

c) $\mathfrak{M}$ induces a tensor automorphism of Res_d (to be denoted by $\mathfrak{M}_d$).

Proof a) The first statement follows from

$$(L_d * Z(\mathcal{G}_1)) \bullet (L_d * Z(\mathcal{G}_2)) \cong (L_d \bullet L_d) * Z(\mathcal{G}_1 * \mathcal{G}_2) \cong L_d * Z(\mathcal{G}_1 * \mathcal{G}_2)$$

and exactness of $\bullet|_{\mathcal{A}_{\underline{c}}}$. In view of (16), in order to check that L_d is a unit object we have only to show that

$$L_d \bullet X \cong X$$

for $X \in \mathcal{A}_d$ (cf the proof of Proposition 2 (b)). It also follows from [L4], 2.9 that

$$L_{d'} \bullet L_d = 0$$

if $d' \neq d$ are different Duflo involutions in $\underline{c}$. Hence $L_{d'} \bullet (L_d * Z(\mathcal{G})) \cong (L_{d'} \bullet L_d) * Z(\mathcal{G}) = 0$, and by exactness of $\bullet|_{\mathcal{A}_{\underline{c}}}$ it follows that $L_{d'} * X = 0$ for $X \in \mathcal{A}_d$. Thus for $X \in \mathcal{A}_d$ we have

$$X \cong \mathbb{I}_{\underline{c}} \bullet X = \bigoplus_{d' \in \underline{c}} L_{d'} \bullet X = L_d \bullet X.$$

This proves (a).

(b), (c) follow immediately from, respectively, properties (iii) and (iv) stated in section 3.2. □

5. Main result

Recall that if $G_1 \supset G_2$ are algebraic groups over a base field k, and $N \in G_1(k)$ is an element which commutes with G_2, then N defines a tensor automorphism of the restriction functor $Res^{G_1}_{G_2} : Rep(G_1) \to Rep(G_2)$. We denote this automorphism by Aut_N.

Theorem 1. *There exists a pair H_d, N_d, where $H_d \subset {}^LG_{\overline{\mathbb{Q}_l}}$ is an algebraic subgroup, and $N_d \in {}^LG(\overline{\mathbb{Q}_l})$ is a unipotent element commuting with N_d; an equivalence of tensor categories $\Phi_d : Rep(H_d) \cong \mathcal{A}_d$, and an isomorphism $Res^{{}^LG}_{H_d} \cong \Phi_d \circ Res_d$, which intertwines the tensor automorphisms Aut_{N_d} and $\mathfrak{M}_d$.*

The pair (H_d, N_d) is unique up to conjugacy.

Remark 5. Rigidity (duality) in $\mathcal{A}_d$ can most probably be interpreted geometrically, and is given by (3) above.

Proof of the Theorem follows directly from Lemma 5 and Proposition 1. □

Below k will denote an algebraically closed field, $\mathrm{char}(k) = 0$. As before, for an algebraic group H over k we will write $\mathcal{R}(H)$ for its representation ring, and set $\mathcal{R}^k(H) = \mathcal{R}(H) \otimes k$. Thus $\mathcal{R}^k(H) = \mathcal{O}(H)^{Ad}$ is the ring of conjugation invariant functions on H. For $s \in H(k)$ we will denote the corresponding character of $\mathcal{R}(H)$ (or of $\mathcal{R}^k(H)$) by $\chi_s : \mathcal{R}(H) \to k$, $\chi_s([V]) = Tr(s, V)$.

Recall the bijection between two-cided cells in W and unipotent conjugacy classes in ${}^LG(k)$ constructed by Lusztig in [L3]. For a two-sided cell $\underline{c}$ we let $N_{\underline{c}} \in {}^LG(\overline{\mathbb{Q}_l})$ denote a unipotent element in the corresponding conjugacy class.

Theorem 2. *For $d \in \underline{c}$ the conjugacy classes of elements N_d and $N_{\underline{c}}$ coincide.*

Proof We will need a characterization of the bijection $\underline{c} \leftrightarrow N_{\underline{c}}$.

We set $J^k_{\underline{c}} = J_{\underline{c}} \otimes_{\mathbb{Z}} k$, $\mathcal{H}^k = \mathcal{H} \otimes k$.

For a unipotent element $N \in {}^LG(k)$ fix a homomorphism $\gamma_N : SL(2) \to {}^LG$ defined over k, such that $N = \gamma(E)$, where $E \in SL(2,k)$ is the standard unipotent element. Let $s^v_{SL(2)} \in SL(2, k[v, v^{-1}])$ be the diagonal matrix with entries v, v^{-1}; and define an element $s^v_N \in {}^LG(k[v, v^{-1}])$ by

$$s^v_N = \gamma_N(s^v_{SL(2)}).$$

Recall the homomorphism $\phi_{\underline{c}} : \mathcal{H} \to J_{\underline{c}} \otimes \mathbb{Z}[v, v^{-1}]$, see (11).

We will also make use of the isomorphism (due to Bernstein)

$$B : \mathcal{R}({}^LG)[v, v^{-1}] \widetilde{\longrightarrow} Z(\mathcal{H}), \tag{17}$$

where $Z(\mathcal{H})$ is the center of $\mathcal{H}$.

Fact 2. [L3] *a) The center of $J^k_{\underline{c}}$, $Z(J^k_{\underline{c}})$ is isomorphic to the algebra $\mathcal{R}^k(Z_{{}^LG}(N_{\underline{c}})) = \mathcal{R}^k(Z_{{}^LG}(\gamma_{N_{\underline{c}}}))$.*

b) The homomorphism $\phi_{\underline{c}}$ sends the center $Z(\mathcal{H})$ into $Z(J_{\underline{c}}) \otimes_{\mathbb{Z}} A$.

c) Let s be a semisimple element of $Z_{{}^LG}(\gamma_{N_{\underline{c}}})$. It defines a character $\chi_s : Z(J_{\underline{c}}) \to k$, and, by extension of scalars, a character $\chi^v_s : J_{\underline{c}} \otimes A \to k[v, v^{-1}]$. The character $\tilde{\chi}_s = \chi^v_s \circ (\phi_{\underline{c}}|_{Z(\mathcal{H})}) : Z(\mathcal{H}) \to k[v, v^{-1}]$ defines a semisimple conjugacy class $\Omega_s \subset {}^LG(\overline{k(v)})$.

Then $\Omega_s \ni s^v_{N_{\underline{c}}} \cdot s$.

Notice that if N, N' are non-conjugate unipotent elements, then $s^v_N \cdot s$ is not ${}^LG(\overline{k(v)})$-conjugate to $s^v_{N'} \cdot s'$ for any $s \in Z_{{}^LG(k)}(N)$, $s' \in Z_{{}^LG(k)}(N')$.

Thus, in view of Fact 2(d), to prove Theorem 2 it suffices to check that setting $k = \overline{\mathbb{Q}_l}$ we get

$$\Omega_s \ni s^v_{N_d} \cdot s_{\mathrm{const}} \tag{18}$$

for some $s_{\mathrm{const}} \in {}^LG(k)$.

The key step in the proof of Theorem 2 is the next Lemma.

We keep notations of Theorem 1. In particular for a Duflo involution $d \in \underline{c}$ we have a subgroup $H_d \subset {}^LG_{\overline{\mathbb{Q}_l}}$ with an equivalence $Rep(H_d) \cong \mathcal{A}_d \subset \mathcal{A}_{\underline{c}}$; thus the Grothendieck group $\mathcal{R}(H_d) = K(\mathcal{A}_d)$ is a subalgebra in $J_{\underline{c}} = K(\mathcal{A}_{\underline{c}})$.

For $V \in Rep({}^LG)$ the nilpotent endomorphism $\log(\mathfrak{M}_d) = \log(N_d)$ yields a filtration (the Jacobson-Morozov-Deligne filtration, see [De1], 1.6, and also [BB], 4.1) on $Res_d(V) \cong \Phi_d(V)$. Let $gr_i(Res^{{}^LG}_{H_d}(V)) \in Rep(H_d)$ denote the i-th associated graded subquotient of this filtration.

Lemma 6. *We have an equality in* $J_{\underline{c}}[v, v^{-1}]$*:*

$$\phi_{\underline{c}}(B([V])) = \sum_d \sum_i v^i [gr_i(Res^{{}^LG}_{H_d}(V))] \tag{19}$$

(where $\phi_{\underline{c}}$ *is as in* (11)*).*

Proof of the Lemma. Let $D^I_{mix} = D^I_{mix}(\mathcal{F}\ell)$ be the $\mathbf{I}$-equivariant l-adic derived category of sheaves on $\mathcal{F}\ell$ with integral Frobenius weights (more presicely, it is a full subcategory of the $\mathbf{I}$-equivariant l-adic derived category, consisting of such complexes that cohomolgy of their stalks carry a Frobenius action with integral weights). Let $\mathcal{P}^I_{mix} \subset D^I_{mix}$ be the full subcategory of perverse sheaves. Then D^I_{mix} is a monoidal category; the central sheaves $Z(\mathcal{G})$ lift to $\mathcal{P}_{mix}$.

We have a standard homomorphism of abelian groups $\tau : K(D^I_{mix}) \to \mathcal{H}$ satisfying

$$\tau([X * Y]) = \tau(X) \cdot \tau(Y).$$

For $V \in Rep({}^LG)$ we have $\tau([Z(V)]) = B([V])$ (see [Ga], 1.2.1).

Then it follows from the definitions that

$$\phi_{\underline{c}}(B[V]) = S(\tau(Z(V) * \bigoplus_d L_d) \mod \mathcal{H}_{<\underline{c}}),$$

where $S : \mathcal{H}_{\leq \underline{c}}/\mathcal{H}_{<\underline{c}} \widetilde{\longrightarrow} J_{\underline{c}} \otimes \mathbb{Z}[v, v^{-1}]$ is the isomorphism sending C_w to t_w.

Any $X \in \mathcal{P}^I_{mix}$ carries the canonical weight filtration (see [BBD]); let $\mathbf{gr}_i(X)$ denote the i-th associated graded subquotient of this filtration. Then $[\mathbf{gr}_i(X)]$ lies in the $\mathbb{Z}$-span of $v^i C_w$.

In particular,

$$S(\tau(\mathbf{gr}_i(Z(V) * L_d)) \mod \mathcal{H}_{<\underline{c}}) \in v^i J_{\underline{c}}.$$

Thus to check (19) it is enough to ensure that the filtration on $\Phi_d(V)$ induced by the weight filtration on $Z(V) * L_d$ coincides with the canonical (Deligne) filtration associated to the nilpotent endomorphism $\log N_{\underline{c}} \in End(\Phi_d(V))$.

We claim that a stronger statement holds. Namely, we claim that the canonical filtration on $Z(V) * L_d$ associated to the logarithm of monodormy coincides with

the weight filtration (this implies the desired statement, as canonical filtration associated to a nilpotent endomorphism is compatible with passing to a Serre quotient category). Since $Z(V) * L_d$ is obtained by nearby cycles from a pure weight zero perverse sheaf (cf [Ga], Proposition 6), the latter statement is a particular case of a general Theorem of Gabber et. al. on coincidence of monodormic and weight filtrations on nearby cycles of a pure sheaf, see [BB], Theorem 5.1.2. □

Remark 6. The proof of Lemma 6 is the only place in this note where the theory of Weil sheaves (rather than the theory of l-adic sheaves on a scheme over $\overline{\mathbb{F}}_q$, which can be safely replaced by the theory of constructible sheaves on the corresponding complex variety) is used.

Corollary 2. *Let $\phi_d : Z(\mathcal{H}) \to \mathcal{R}(H_d) \otimes A \subset J_{\underline{c}} \otimes A$ be the homomorphism given by $z \mapsto z * t_d$. For a semisimple $s \in H_d(k)$ consider the character $\chi_s^v : \mathcal{R}(H_d) \to k[v, v^{-1}]$ obtained from $\chi_s : \mathcal{R}(H_d) \to k$ by extension of scalars. The character $\tilde{\chi}_s = \chi_s^v \circ (\phi_d) : Z(\mathcal{H}) = \mathcal{R}(^LG) \to k[v, v^{-1}]$ defines a semisimple conjugacy class in $^LG(\overline{k(v)})$.*

This conjugacy class contains element $s^v_{N_d} \cdot s$. □

Proof of Theorem 2. Let M be an irreducible $J^k_{\underline{c}}$ module on which the idempotent $t_d \in J_{\underline{c}}$ acts by a nonzero operator. The commutative subalgebra $K(\mathcal{A}_d)^k = \mathcal{R}^k(H_d) \subset t_d J^k_{\underline{c}} t_d$ acts on the finite dimensional vector space $t_d M$, with unit element t_d acting by identity; let $u \in t_d M$ be an eigen-vector. The corresponding character $\chi_u : \mathcal{R}(H_d) \to k$ comes from an element $s_u \in H_d(k)$; by Corollary 2 the center $Z(\mathcal{H})$ acts on the vector $u \in \phi^*_{\underline{c}}(M)$ by the character $\chi_{s_u \cdot s^v_{N_d}}$, which shows (18). □

5.1. We now restrict attention to the unique Duflo involution in $\underline{c} \cap W^f$; we call it d^f.

Theorem 3. *a) The set of semisimple objects of $\mathcal{A}_{d^f}$ is $\{L_w \mid w \in W^f \cap \underline{c}\}$.*
b) H_{d^f} contains a maximal reductive subgroup of the centralizer $Z_{^LG}(N_d)$.

Remark 7. A statement stronger than that of Theorem 3(b) follows from results of [AB] (see section 6 below for details): we show there that in fact $H_{d^f} = Z_{^LG}(N_{\underline{c}})$. The argument in [AB] does not use the “nonelementary” result about asymptotic Hecke algebras cited in Fact 3 below (the proof of this result in [L3] relies on the theory of character sheaves).

Proof of the Theorem. The set of irreducible objects of $\mathcal{A}_d$ consists of those L_w, which are subquotients of $Z(V) * L_d$ for some $V \in Rep(^LG)$. Identifying $J_{\underline{c}} = K(\mathcal{A}_{\underline{c}})$ we get $[Z(V) * L_d] = B(V) * t_d|_{v=1}$; since $B(V) * t_d = t_d * B(V) \in t_{d^f} \cdot J_{\underline{c}} \cdot t_{d^f} = J^f_{\underline{c}}$ we see that indeed any subquotient of $Z(V) * L_d$ has the form L_w, $w \in \underline{c} \cap W^f$.

To check that $L_w \in \mathcal{A}_{d^f}$ for all $w \in W^f \cap \underline{c}$, it suffices to check that for any proper subset $S \subsetneq \underline{c} \cap W^f$ there exists V such that $B(V) * t_d$ does not lie in the span of t_w, $w \in S$. This follows from the next Lemma.

Let $\phi^f_{\underline{c}} : Z(\mathcal{H}) \to J^f_{\underline{c}} \otimes A$ be the homomorphism $z \mapsto z * t_{d^f}$.

Lemma 7. *For any proper subalgebra $J' \subset (J^f_{\underline{c}})^k$ we have $im(\phi^f_{\underline{c}}) \not\subset J' \otimes k(v)$.*

We will deduce the Lemma from another result of Lusztig (see Remark 7 above).

Fact 3. *Let $J^f_{\underline{c}} \subset J_{\underline{c}}$ be the span of $\{t_w \mid w \in W^f \cap \underline{c}\}$.*

Then $J^f_{\underline{c}} = t_{df} \cdot J^f_{\underline{c}} \cdot t_{df}$ is a subalgebra.

The map $z \mapsto z \bullet t_{df}$ induces an isomorphism between $Z(J_{\underline{c}}) \otimes k$ and $J^f_{\underline{c}} \otimes k$ (recall that k is a characteristic zero field).

Proof of Lemma 7. Comparing Fact 3 and Fact 2(a) we can identify $(J^f_{\underline{c}}) \otimes k = \mathcal{R}^k(Z_{^LG}(N_{\underline{c}})) = \mathcal{R}^k(Z_{^LG}(\gamma_{N_{\underline{c}}})$.

If the image of $\phi^f_{\underline{c}}$ is contained in $J' \otimes k[v, v^{-1}]$ for $J' \subsetneq (J^f_{\underline{c}}) \otimes k$, then for some non-conjugate semi-simple elements $s_1, s_2 \in Z_{^LG(k)}(N_{\underline{c}})$ we have $\chi_{s_1} \circ \phi^f_{\underline{c}} = \chi_{s_2} \circ \phi^f_{\underline{c}}$. This means that $s_1 \cdot s^v_{N_{\underline{c}}}$ is conjugate to $s_2 \cdot s^v_{N_{\underline{c}}}$ in ${}^LG(\overline{k(v)})$; this is well-known to be impossible. □

Proof of Theorem 3(b). Comparing Facts 2 and 3 we see that there exists an isomorphism $I_{lus} : (J^f_{\underline{c}}) \otimes k \widetilde{\longrightarrow} \mathcal{R}^k(Z_{^LG}(N_{\underline{c}}))$, such that for $s \in Z_{^LG(k)}(N_{\underline{c}})$ we have $\chi^v_s \circ I_{lus} \circ \phi_{df} = \chi_{s^v_{N_{df}}}$.

By part (a) of the Theorem we have also an isomorphism $I_{gaitsg} : J^f_{\underline{c}} \widetilde{\longrightarrow} \mathcal{R}(H_d)$, and, by Corollary 2 for $s \in H_d(k)$ we have $\chi^v_s \circ I_{gaitsg} \circ \phi_{df} = \chi_{s^v_{N_{df}}}$.

The proof of (a) shows that for two different characters $\chi_1, \chi_2 : J^f_{\underline{c}} \to k$ we have $\chi_1 \circ \phi_{df} \neq \chi_2 \circ \phi_{df}$; hence for $s \in H^{ss}_d$ we have $\chi_s \circ I_{lus} = \chi_s \cdot I_{gaitsg}$, i.e. the isomorphism $I_{gaitsg} \otimes k \circ I^{-1}_{lus} : \mathcal{R}^k(Z(N_{\underline{c}})) \to \mathcal{R}^k(H_d)$ coincides with the natural restriction map. Thus statement (b) of the Theorem follows from the next Lemma.

Lemma 8. *If a homomorphism of algebraic groups over a field of chracteristic zero $i : H_1 \to H_2$ induces an isomorphism of varieties $H_1/\mathrm{Ad}\, H_1 \widetilde{\longrightarrow} H_2/\mathrm{Ad}\, H_2$ then it induces an isomorphism of maximal reductive quotients $H^{red}_1 \widetilde{\longrightarrow} H^{red}_2$.*

Proof We can assume that the base field is algebraically closed.

Since $H/\mathrm{Ad}\, H = H^{red}/\mathrm{Ad}\, H^{red}$ for an algebraic group H, we can replace H_1, H_2 by H^{red}_1, H^{red}_2, and assume that H_1, H_2 are reductive. It is clear that i is injective (otherwise i sends a nontrivial semisimple conjugacy class in H_1 to identity of H_2).

Let us first check that i induces an isomorphism of connected components of identity $H^0_1 \longrightarrow H^0_2$.

It is easy to see that for a (not necessarily connected) reductive group H the connected component of identity in $H/\mathrm{Ad}\, H$ is described by $(H/\mathrm{Ad}\,(H))^0 = T/N_H(T)$. Hence H_1 and H_2 have common Cartan T, and the image of the map $N_{H_i}(T)/T \to \mathrm{Aut}\,(T)$ does not depend on $i = 1, 2$. It is enough to check that the image of $N_{H^0_i}(T)/T$ in $\mathrm{Aut}\,(T)$ does not depend on i either.

So take $x \in N_{H_1}(T)$. It is easy to see that $\mathrm{Ad}\, x|_T \in im(N_{H^0_i}(T)/T)$ iff $\mathrm{Ad}\, x|_{H^0_i}$ is an inner automorphism of H^0_i. However the latter condition is equivalent to $\dim((H_i/\mathrm{Ad}\, H_i)^x) = \dim((H_i/\mathrm{Ad}\, H_i)^0)$, where $(H_i/\mathrm{Ad}\, H_i)^x$ is the connected component of the class of x. (This is so because $\dim(\mathfrak{g}^F)$, where F is a generic automorphism of a reductive Lie algebra $\mathfrak{g}$ in a fixed coset modulo inner automorphisms, is maximal when the coset is trivial). Since $\dim((H_1/\mathrm{Ad}\, H_1)^x) = \dim((H_2/\mathrm{Ad}\, H_2)^x)$ we get the statement.

To finish the proof it remains to see that i induces a bijection on the set of connected components, $i : H_1/H^0_1 \widetilde{\longrightarrow} H_2/H^0_2$. For this it is enough to prove the statement of the Lemma for finite groups H_1, H_2. Then it is known as Jordan's Lemma, see e.g. [Se], Lemma 4.6.1. □

Theorem 3 is proved. □

Corollary 3. *The semisimple monoidal category $A^f_{\underline{c}}$, whose set of irreducible objects is $\{L_w \,|\, w \in \underline{c} \cap W^f\}$, and the monoidal structure is provided by truncated convolution (see* [L4]*), is equivalent to the category of representations of $Z^{red}_{{}^LG}(N_{\underline{c}})$, the maximal reductive quotient of $Z_{{}^LG}(N_{\underline{c}})$.*

Proof Theorem 3(a) implies that $A^f_{\underline{c}}$ is the category of semisimple objects in $\mathcal{A}_{d^f}$. The latter is identified with $Rep(H_{d^f})$ by Theorem 1, thus $A^f_{\underline{c}} \cong Rep(H^{red}_{d^f})$. In view of Theorem 3(b) we have $H^{red}_{d^f} = Z^{red}_{{}^LG}(N_{d^f})$, and by Theorem 2 N_{d^f} is conjugate to $N_{\underline{c}}$. □

The statement of the Corollary was conjectured in [L4], §3.2.

6. Announcement of some further results

In this final section we describe results related to the subject of the present note, to appear in [AB].

Set $\mathcal{P}^f$ denote the Serre quotient category $\mathcal{P}/P_{W-W^f}$ (notations of 3.3); thus the irreducible objects of $\mathcal{P}^f$ are indexed by minimal length representatives of cosets $W_f\backslash W/W_f$ (which are in bijection with dominant coweights of G). $\mathcal{P}^f$ carries a filtration by Serre subcategories $\mathcal{P}^f_{\leq\underline{c}} = \mathcal{P}^f \cap \mathcal{P}_{\leq\underline{c}}$. Above we constructed an imbedding of the category $Rep(H_{\underline{c}})$ in the Serre subquotient category $\mathcal{P}^f_{\leq\underline{c}}/\mathcal{P}^f_{<\underline{c}}$ for a subgroup $H_{\underline{c}} = H_{d^f} \subset Z_{{}^LG}(N_{\underline{c}})$, and stated without proof (see Remark 7) that $H_{\underline{c}} = Z_{{}^LG}(N_{\underline{c}})$ (though checked that $H^{red}_{\underline{c}} = Z_{{}^LG}(N_{\underline{c}})^{red}$).

Let $\mathcal{N} \subset {}^LG$ be the subvariety of unipotent elements, and $Coh^{{}^LG}(\mathcal{N})$ be the category of ${}^LG$-equivariant coherent sheaves on $\mathcal{N}$. For $N \in \mathcal{N}$ let $Coh^{{}^LG}_{\leq N}(\mathcal{N}) \subset Coh^{{}^LG}(\mathcal{N})$ be the subcategory of sheaves whose support is contained (as a set, not necessarily as a scheme) in the closure of the orbit ${}^LG(N)$, and let $Coh^{{}^LG}_{<N}(\mathcal{N}) \subset Coh^{{}^LG}_{\leq N}(\mathcal{N})$ consist of those sheaves whose restriction to ${}^LG(N)$ iz zero. Then the Serre quotient $Coh^{{}^LG}_{\leq N}(\mathcal{N})/Coh^{{}^LG}_{<N}(\mathcal{N})$ is identified with the category of equivariant sheaves on the formal neighborhood of ${}^LG(N)$ in $\mathcal{N}$; it contains $Coh^{{}^LG}({}^LG(N)) = Rep(Z_{{}^LG}(N))$ as a full subcategory.

Situations described in the two previous paragraphs look similar; this suggests a relation bewteen $\mathcal{P}^f$ and $Coh^{{}^LG}(\mathcal{N})$. Indeed, we show in [AB] that such a relation exists on the level of derived categories. To formulate a precise result we need another notation. Let ${}^L\mathfrak{h}$ be the (abstract) Cartan subalgebra in the Lie algebra ${}^L\mathfrak{g}$ of ${}^LG$, and let ${}^L\mathfrak{h}_0$ be the scheme of finite length $|W_f|$ defined by ${}^L\mathfrak{h}_0 = {}^L\mathfrak{h} \times_{{}^L\mathfrak{h}/W_f} \{0\}$ (thus ${}^L\mathfrak{h}_0$ is the spectrum of the cohomology ring of the flag variety G/B). In [AB] we construct an equivalence of triangulated categories

$$D^b(\mathcal{P}^f) \cong D^b(Coh^{{}^LG}(\mathcal{N} \times {}^L\mathfrak{h}_0)), \tag{20}$$

where ${}^LG$ acts trivially on ${}^L\mathfrak{h}_0$. Equivalence (20) is compatible with one in Theorem 1 (for $d = d^f \in W^f$) in the natural sense; this allows, in particular, to show that $H_{d^f} = Z_{{}^LG}(N_{\underline{c}})$.

The tautological t-structure on $D^b(\mathcal{P}^f)$ defines, by means of (20), a t-structure on $D^b(Coh^{{}^LG}(\mathcal{N} \times {}^L\mathfrak{h}_0))$ (and also on $D^b(Coh^{{}^LG}(\mathcal{N})))$. This t-structure coincides with the perverse t-structure on equivariant coherent sheaves, corresponding to the middle perversity, see Example 1 at the end of [B].

Along with the description (20) of the category $D^b(\mathcal{P}^f)$, we give a simliar description of the "larger" category $D^b(\mathcal{P})$. To state we need some notations. Let ${}^L\mathfrak{g}$ be the Lie algebra of ${}^LG$, and let $\tilde{\mathcal{N}} = T^*({}^LG/{}^LB) \to \mathcal{N}$, ${}^L\tilde{\mathfrak{g}} \to {}^L\mathfrak{g}$ be the Spinger-Grothendieck maps. Set $\mathbf{St} = {}^L\tilde{\mathfrak{g}} \times_{{}^L\mathfrak{g}} \tilde{\mathcal{N}}$; thus $\mathbf{St}$ is a nonreduced scheme, whose reduced part is the "Steinberg variety of triples" $\mathbf{St}^{red} = St = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ of ${}^LG$. We construct an equivalence

$$D^b(\mathcal{P}) \cong D^b(Coh^{{}^LG}(\mathbf{St})). \tag{21}$$

The isomorphism of Grothendieck groups underlying (21) is the well-known isomorphism between two different realization of the group ring $\mathbb{Z}[W]$, see e.g. [CG], 7.2. (A "graded" version of (21), in the sense of [BGS], 4.3, which relates the derived category of mixed sheaves to the derived category of ${}^LG \times G_m$ equivariant coherent sheaves on $\mathbf{St}$ gives the isomorphism between two different geometric realizations of the affine Hecke algebra $\mathcal{H}$, see [CG], chapter 7; and also *loc. cit.*, Introduction, p. 15.)

The natural proper morphism $\pi : \mathbf{St} \to \mathcal{N} \times {}^L\mathfrak{h}_0$ yields a functor $\pi_* : D^b(Coh^{{}^LG}(\mathbf{St})) \to D^b(Coh^{{}^LG}(\mathcal{N} \times {}^L\mathfrak{h}_0))$; under the equivalences (20), (21) this functor corresponds to the tautological projection functor $D^b(\mathcal{P}) \to D^b(\mathcal{P}^f)$.

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